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THE NUMBER OF ROUNDS NEEDED TO EXCHANGE INFORMATION WITHIN A GRAPH

Susan F. ASSMANN and Daniel J. KLEITMAN
Massachusetts Institute of Technology, Department of Mathematics, Cambridge, MA 02139, USA

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In [3] Cederbaum proposes the problem of finding a labelling of a connected graph G which minimizes the number of rounds needed to exchange information between all the points, where a round consists of the vertices of G each broadcasting once in order of their labels. In this paper we give a complete solution to the problem.

Introduction

In [3], Cederbaum poses the following question. Suppose we are given a labelled connected graph G with n vertices, such that at time $t=0$ each vertex has some item of information known only to itself. At integral time steps $t=1,2,3,\ldots$ vertex $v=t \pmod n$ broadcasts all the information it knows to all vertices adjacent to it. These vertices then update their information with this message. A *round* consists of vertices 1 through n each broadcasting once (in that order). We wish to find a relabelling of G which minimizes the number of rounds (including perhaps a partial final round) needed before each vertex knows all the information. Call this minimum number of rounds $c(G)$.

Cederbaum's problem is in some ways similar to several versions of the 'gossip problem'. See [1], [2], [4], and [6]-[10]. In this problem also, each vertex of a graph G with n points has some piece of information known only to itself. The allowed communications are 'telephone calls' between two vertices adjacent in G, during which the two vertices share all the information they have at the time. The question is how many calls are needed before all vertices know everything, and what order of calls achieves this minimum. If G contains a quadrilateral, the minimum is $2n-4$; if not, and $n\geq 4$, the minimum is $2n-3$. If only 'one-way calls' or 'letters' are allowed, and G is replaced by a strongly connected digraph D with n points, the minimum is $2n-2$ calls. The gossip problem with G the complete graph on n points and communication by k-party calls has also been solved.

Cederbaum gives several partial results for the question of [3]:

(1) $c(G)=1$ iff G contains a vertex of degree $n-1$.

(2) If G is a tree, $c(G) = \mathrm{rad}(G)$, the radius of G. Furthermore, for any graph, $c(G) \le \mathrm{rad}(G)$.

(3) If G is a cycle, $n > 3$, then $c(G) = 2$. Furthermore, for any Hamiltonian graph, $c(G) \le 2$.

In this paper, we give a complete solution to the problem. From now on, we assume that G has no vertices of degree $n-1$.

We first show that only two rounds are required if G is 2-connected. Next we define a 'cliquing' operation on graphs, and show that a certain class of graphs can be obtained from trees by performing this operation a finite number of times. We show that any graph G in this class satisfies $c(G) = \mathrm{rad}(G)$. Finally, given any graph G, connected but not 2-connected, we define an associated graph G', based on the 2-connected components of G, which is in this class and satisfies $c(G) = c(G')$. Then $c(G)$ will equal $\mathrm{rad}(G')$.

Results

Theorem 1. *If G is 2-connected and has no vertex of degree $n-1$, then $c(G) = 2$. Note that this generalizes Cederbaum's third result.*

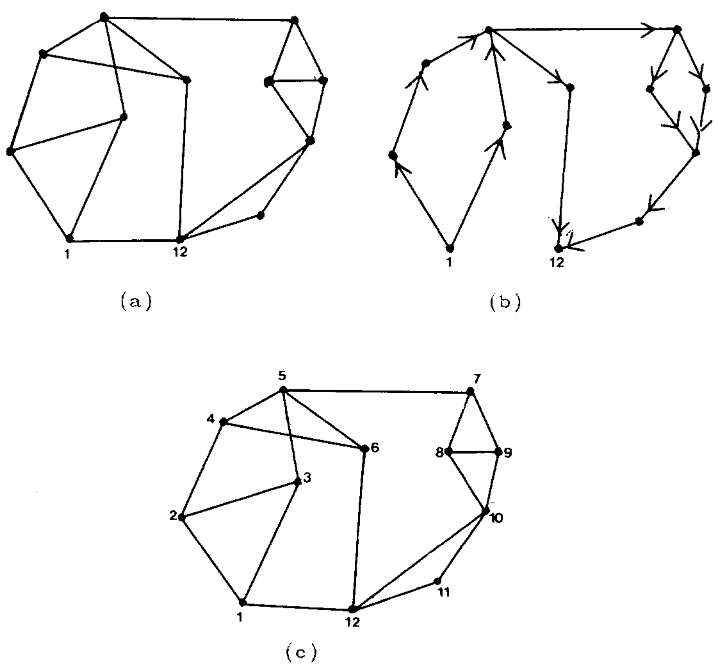
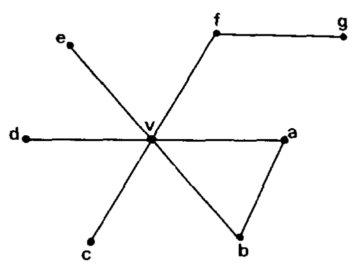


Fig. 1. An optimal labelling of a 2-connected graph. (a) The graph with first and last vertices chosen. (b) A collection of directed paths. (c) An optimal labelling.

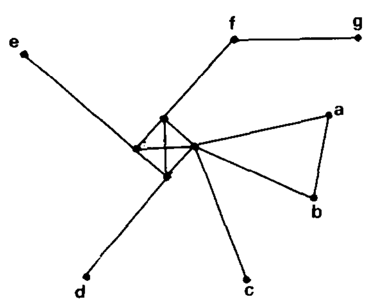
Proof. Pick any two adjacent vertices in G and label them 1 and n. Because G is 2-connected, we can find a collection of compatible directed paths from 1 to n which does not include the edge $\{1, n\}$ and which includes every vertex in G. See Figs. 1(a) and 1(b). We can assign labels to the remaining vertices in such a way that each edge of each directed path goes from a lower-labelled vertex to a higher-labelled one. See Fig. 1(c). Note that in general neither the collection of paths nor the optimal labelling will be unique.

As vertices 1 through $n-1$ broadcast in turn, information moves along all these paths and converges at vertex n. When vertex n broadcasts for the first time, it knows everything and transmits this information to vertex 1. During the second round, the information spreads from vertex 1 along all the paths, and thus reaches every vertex.

Consider the following 'cliquing' operation on a graph G. Choose a vertex v of G which has at least three pairwise non-adjacent neighbors. Partition the set of neighbors of v into nonempty sets $W_1, W_2, \ldots, W_k$, with $3 \le k$, in such a way that there is no edge from a point in one of these sets to a point in one of the others. Replace v by a clique K of size k, and attach each vertex of W_i to the ith vertex of K, for $1 \le i \le k$. See Fig. 2.



(a)



(b)

Fig. 2. An illustration of the cliquing process. (a) The graph G. (b) The resulting graph G', where the partition of neighbors of v was chosen to be $\{a, b, c\}$, $\{d\}$, $\{e\}$, $\{f\}$.

Lemma 1. *Let G be a connected graph such that every cycle of G lies in some clique, and every point in a clique of size at least three has some neighbor not in the clique. Suppose there are m maximal cliques of size at least three. Then G can be obtained from some tree by m applications of the cliquing process.*

Proof. By induction on m.

If $m=0$, G itself is a tree, so the lemma holds.

Suppose it holds for $m=r$, and G has $r+1$ maximal cliques. Pick any maximal clique $K=\{v_1,\dots,v_k\}$ of G, and let the neighborhood of v_i, not counting points in K, be called W_i, for $1\le i\le k$.

For each $1\le i<j\le k$, the intersection of W_i and W_j is empty. Suppose not. Without loss of generality, assume there is some w in both W_1 and W_2. For $3\le i\le k$, if w is not in W_i, then v_1, w, v_2, v_i is a cycle which does not lie in any clique. But if w is in W_i for all i, $1\le i\le k$, then K was not a maximal clique. In either case we reach a contradiction, so our claim is proved.

Also, there is no edge from w_i in W_i to w_j in W_j for $i\ne j$. If there were, v_i, w_i, w_j, v_j would be a cycle that did not lie in any clique.

Now, let H be the graph obtained by contracting the clique K to a single point v. It is clear from the above two claims that G can be obtained from H by one application of the cliquing process. H also satisfies the hypotheses of the lemma, and has only r maximal cliques, so it can be obtained from some tree by r applications of the process. Putting these two facts together completes the proof of the lemma.

Lemma 2. *Suppose G is a graph obtained from a tree T by performing the above operation m times* ($m\ge 0$). *Then* $c(G)=\mathrm{rad}(G)$.

Proof. By induction on m.

If $m=0$, then $G=T$, and by Cederbaum's second result, $c(G)=\mathrm{rad}(G)$.

Suppose that for any tree T the lemma is true for $m\le r$. We must show that it is true for $m=r+1$. Suppose

$$T\to G_1\to G_2\to\cdots\to G_r\to G_{r+1}=G$$

is the series of transformations which produces G from T. Let $H=G_r$. By the inductive hypothesis $c(H)=\mathrm{rad}(H)$. Clearly the process does not decrease either the radius or the number of rounds needed.

If $\mathrm{rad}(G)=\mathrm{rad}(H)$ we have $c(H)\le c(G)\le \mathrm{rad}(G)\le \mathrm{rad}(H)=c(H)$ so $c(G)=\mathrm{rad}(G)$.

Suppose $\mathrm{rad}(G)>\mathrm{rad}(H)$. The only way this could happen is if v, the vertex changed to a clique K in the transformation from H to G, is on some longest path through H (say, a, v, b is a segment of such a path) and a and b are attached to different vertices of K. Let v_1 and v_2 be the vertices of K which a and b (resp.) are attached to in G. Consider the graph G' obtained from G by deleting the other vertices of K and all components which this disconnects from the component containing v_1. Clearly $c(G') \le c(G)$ and $\mathrm{rad}(G') = \mathrm{rad}(G)$. By Lemma 1, G' is a graph obtained from some tree by no more than r applications of our process, so by our inductive hypothesis $c(G') = \mathrm{rad}(G')$. Thus $c(G') \le c(G) \le \mathrm{rad}(G) = \mathrm{rad}(G') = c(G')$, so $c(G) = \mathrm{rad}(G)$, and we are done. See Fig. 3.

Now, suppose we are given a graph G, connected but not 2-connected, with no vertex of degree $n-1$. Associate with it the following graph G'.

Vertices: One vertex for each cut point of G. These are called type 1 vertices. One vertex for each block (2-connected component) of G which contains only one cut point. (We will call such blocks *leaf blocks*, because they are the leaves in the block-cutpoint tree of G. See Harary [5].) These are called type 2 vertices. One additional vertex for each leaf block which includes a point at distance 2 from the cut point. These are called type 3 vertices.

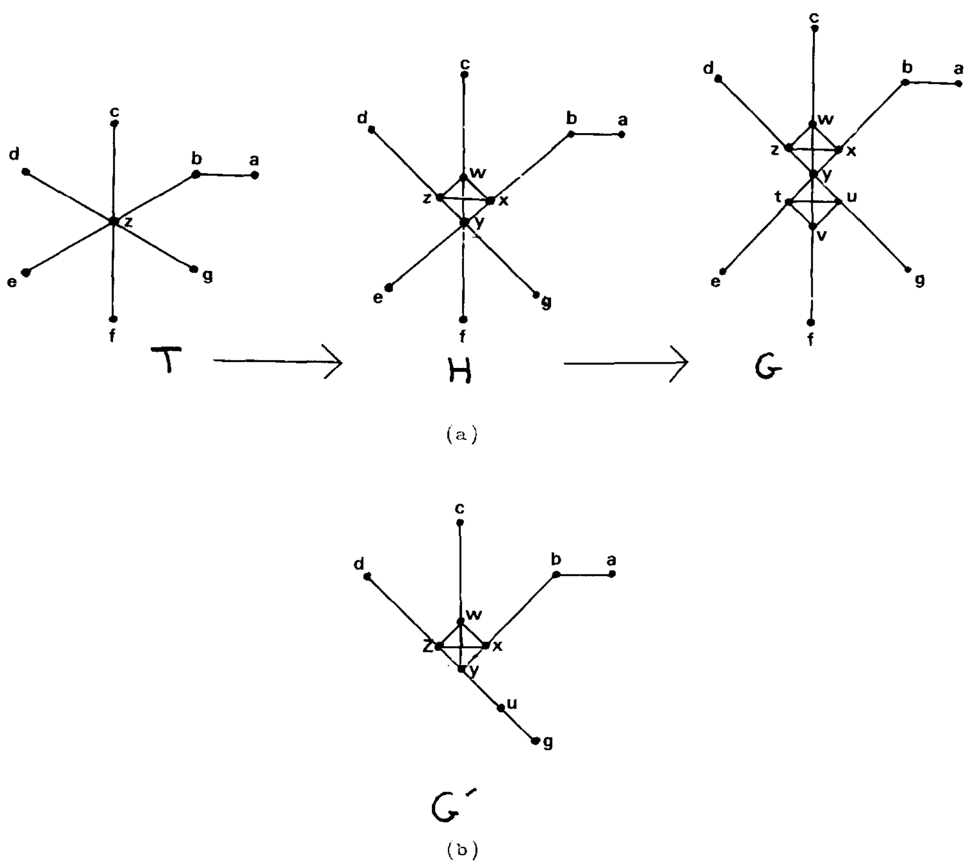


Fig. 3. An illustration of the proof of Lemma 2. (a) The series of transformations from T to G. (b) The graph G', where a, b, x, y, g was the chosen longest path in H.

Edges: Two type 1 vertices are adjacent iff they lie in a common block. Each type 2 vertex is connected to the cut point of its associated leaf block. Each type 3 vertex is connected to the type 2 vertex associated with the same block.

See Fig. 4. Note that if G is a tree, $G'=G$.

Lemma 3. *G' will be a graph satisfying the hypotheses of Lemma* 1.

Proof. Clearly G' will be connected. Suppose $v_1, v_2, \ldots, v_k$ is a cycle in G'. By the construction of the graph, each v_i must be a cut point of G, rather than a type 2 or type 3 vertex. Also, v_i and v_{i+1} lie in a common block in G, for $1 \le i \le k-1$, and v_1 and v_k also lie in a common block. If these blocks are not all the same, we have a ring of 2-connected components, which is impossible. Thus, there is a single block B containing all the vertices v_1 through v_k, and hence they form a clique in G'. Also, each v_i is adjacent in G' to a type 1 vertex not in B or to a type 2 vertex. Thus every point in a clique of size at least three has some neighbor not in the clique.

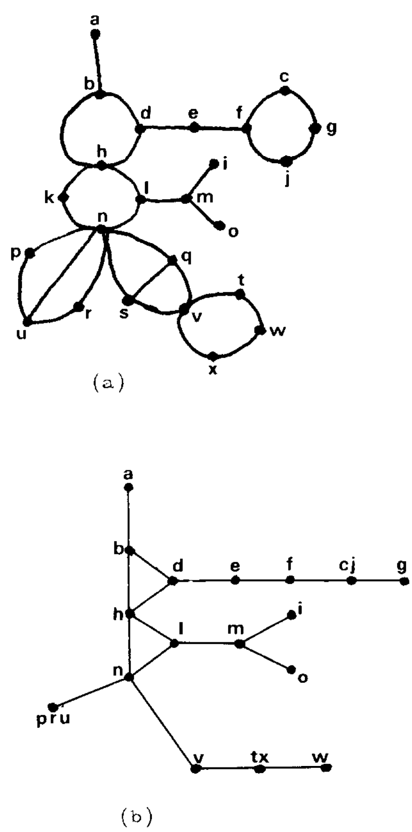


Fig. 4. Finding the associated graph G' for a graph G. (a) The graph G. (b) The associated graph G'. Vertex h is the center, and the radius is 5.

Theorem 2. *If G is a graph, connected but not 2-connected, with no vertex of degree $n-1$, then $c(G)=\mathrm{rad}(G')$.*

Proof. First we show that $\mathrm{rad}(G')$ rounds suffice.

Pick a center vertex of G'. It will necessarily be a type 1 vertex. Label this vertex n. It will either lie in the center C of the block-cutpoint tree, if this is a block, or it will be the center of the tree. In this case, let C be one of the blocks which n lies in.

Label the blocks which are farthest from C (in the block cut-point tree) as in the proof of Theorem 1, with the cut vertex chosen as the last vertex in the block. However, do not actually label this last vertex (the cut vertex).

Carry out the same procedure on the components next closest to C, and so on, each time choosing the cut-point nearest C to be the last vertex in the block, but not labelling it.

Finally label component C in the same way, with vertex n as the last vertex. See Fig. 5.

After one round, each cut vertex knows all the information from the blocks 'outer' to it in the block-cutpoint tree. In particular, vertex n knows everything. Furthermore, when vertex n broadcasts at the end of the first round, it starts the total message on its way in each of the blocks n lies in.

At the end of the second round, this information has circulated throughout these blocks and has reached the first vertices of the blocks next further out.

Continuing in this fashion, a type 1 vertex v at distance i from n in G' knows the total information by the end of round $i+1$, and so do all vertices in blocks inner to it.

A vertex v at distance 1 from the cutpoint of a leaf block learns the information by time i, where i is the distance (in G') between the type 2 vertex of this block and vertex n. This is because the cut point that v is adjacent to is at distance $i-1$ from n in G'. This cut point learns and broadcasts the information by round i, informing v.

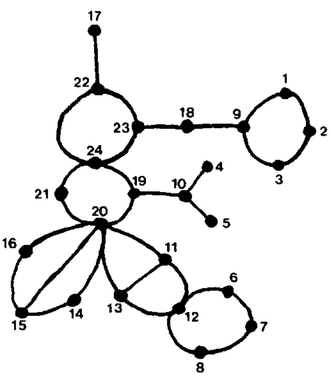


Fig. 5. An optimal labelling for the graph of Fig. 4(a).

If a leaf block has a vertex v at distance more than one from its cut point, v learns the information by the end of round i, where i is the distance from the type 3 vertex for this block to vertex n in G'. This because the vertex with the lowest label in the block learns the information by time $i-1$, and during the next round spreads it throughout the block.

Since each point at the maximum distance from n in G' is either a type 2 or type 3 vertex, this completes the proof that $\mathrm{rad}(G')$ rounds suffice.

Next, we must show that $\mathrm{rad}(G')$ rounds are necessary.

The cut points of G are the only channel through which information can be exchanged between blocks. Also, every path in G which connects two cut points either lies within a single block or goes through a third cut point. Points at distance 1 from the cut points of leaf blocks can only communicate with the rest of G through the cut vertices of their blocks. Finally, all information passing between the rest of G and points in leaf blocks which are at distance 2 or more from their cut points must pass through some point in the same block at distance 1 from the cut point. These constraints on information flow are mirrored in the construction of the graph G'. Since information must be exchanged among all these points, at least $c(G')$ rounds are necessary. Since G' is a graph of the type described in Lemma 1, $c(G')=\mathrm{rad}(G')$, so at least $\mathrm{rad}(G')$ rounds are necessary. This completes the proof.

In summary, then,

$$c(G)=\begin{cases}1 & \text{if there is a vertex of degree } n-1,\\ 2 & \text{if there is no such vertex, but } G \text{ is 2-connected,}\\ \mathrm{rad}(G') & \text{otherwise.}\end{cases}$$

Acknowledgements

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